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CONTINUAL MECHANOCHEMICAL MODEL OF MUSCULAR TISSUE

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The behavior of active muscular tissue is described with the help of a closed system of equations of motion of a two-phase, multicomponent, anisotropic continuous medium, with the mechanochemical processes occurring within it taken into account. The fundamental hypotheses are based on the information of general character concerning the structure and performance of the muscular tissue. It is assumed that the phase in which the mechanochemical reactions take place is viscoelastic, while the other phase is assumed elastic. The medium is assumed to have single velocity, although a passage of components between the phases is allowed. The laws of conservation are given and the rheological equations are written in accordance with the general principles of the mechanics of continuous medium and thermodynamics of irreversible processes [1 – 4]. It is shown that the model constructed describes, e.g., such characteristic properties of the muscle tissue as the existence of stresses in the absence of strains, zero-load deformations, and dissipation of energy in the state of mechanical equilibrium.

The activity of the muscular tissue is governed by chemical processes taking place in the tissue, within the specific ordered structures called myofibrillae and, in the final count, by the mechanochemical reactions which affect the form or the relative distribution of the protein molecules [5 – 8]. Outside the myofibrillae we have various auxilliary systems, the connecting tissue and other structures, including capillary blood vessels which serve as the source of initial chemical compounds. The onset of active muscular contraction is connected with the arrival of specific reagents at the myofibrillae.

The study of various physiological phenomena (such as the working of the heart, propagation of an excitation in the tissues, organization of movements, regulation of the blood circulation and breathing) requires, on the one hand, a rheological equation of the active muscle to be available, on the other hand it requires the knowledge of the relation connecting the basic mechanical parameters of the muscle (such as length, load, etc.) with the parameters defined by the chemical processes, such as the energy needed by the muscle during contraction. The absence of a sufficiently general model of the muscle makes necessary the use, at the present time, of much simpler models [9 – 11].

The following problems are usually considered in the literature on their own: the propagation of excitation throughout the muscle tissue, mechanics of the muscle as a whole in terms of the load – length, the energetics and biochemistry of the muscular activity.

All existing models of the muscular tissue are, with the exception of one given in [12], one-dimensional and are given, as a rule, in the form of relations connecting the load and extension and their derivatives with respect to time, with certain additional parameters which have the dimensions of the load or extension. These additional parameters have no explicit connection with the chemical or other internal processes taking place within the muscle, and their variation with respect to time is defined a priori, differently for the muscle in the active and in the passive state [12 – 18]. Thus the author of [12] proposes a rheological equation of the form

$$p_{ij}{}^{d} = k_{ijln}(\varepsilon_{ln} - N_{ln}{}') \tag{0.1}$$

where $p_{ij}{}^{d}$ is the deviator of the stress tensor, ε_{ij} is the strain tensor and $N_{ij}{}'$ is a tensor parameter ("biofactor") characterizing the activity of the tissue; $N_{ij}{}' = 0$ and $N_{ij}{}' \neq 0$ for the passive and the active state, respectively. The so-called three-element models (see e.g. [19]) contain practically the same additional parameters.

Another group of one-dimensional models explains the meaning of the additional parameters contained in the rheological equation using the terms and quantities chatacterizing the microstructure of the muscle and its hypothetic "internal" mechanics (see e.g. [16 – 18, 20, 21]). We have already noted that these models are also based on the assumption that active deformations (contractions), i.e. zero-stress deformations, are possible.

The purpose of this paper is to construct a model of a continuous medium with properties typical of a muscular tissue, without assumptions of existence of active deformations and without using the discussion concerning the microstructure of the myofibrillae and the processes taking place within them.

1. Basic equations. The initial assertions are as follows [5 – 7].

1°. A muscle performs work by means of direct transformation of energy liberated during the mechanochemical processes.

2°. The mechanochemical reactions take place within a large number of small finite regions distributed throughout the whole volume of the muscle.

3°. The sources of the initial chemical reagents are also distributed throughout the whole volume of the muscle.

4°. The muscle tissue is anisotropic and possesses elastic and viscous properties.

The viscous properties depend mainly on the myofibrillae while the elastic properties are governed by the connecting tissue and other structures.

In accordance with the above we shall consider a two-phase continuous medium, each phase containing n components. Phase 1 consists of the myofibrillae, and phase 2 of all the remaining structures. The phases, in principle, differ from each other by the fact that mechanochemical reactions can only take place in one of them (in phase 2). The elementary volume v of the continuous medium is equal to the sum of the volumes v^1 and v^2 occupied by the first and second phase, respectively. The phases interface is assumed to be permeable for certain components.

The construction of equations for a two-phase medium involves averaging various quantities over the area and over the volume. This gives rise to additional parameters, namely the porosity $m = v^1 / v$ and permeability (relative area of cross section pertaining to phase 1). In the present work we assume that the medium is statistically homogeneous, hence permeability is equal to porosity. The equations are written for the averaged parameters. The parameters obtained by averaging over the volume of one of the phases (v^1 or v^2) are contained within the angular brackets, while the symbols without the brackets denote the parameters obtained by averaging over the total elementary volume v. The upper indices denote the phase and the lower (nontensorial) indices denote either the given component or the given reaction.

Let us define the density of the kth component in the ith phase by

$$\rho_k{}^i = m_k{}^i / v$$

where $m_k{}^i$ is the mass of the kth component in the ith phase contained in the elementary volume v, and let us use this to define the mass concentration

$$c_k{}^i = \rho_k{}^i / \rho \qquad (k = 1 \ldots n) \tag{1.1}$$

where ρ is the mean density of the mixture. We assume that the density ρ of the medium and the true phase densities $\langle \rho^i \rangle$ averaged over the respective phases (true densities) are constant and equal to each other

$$\rho = \langle \rho^1 \rangle = \langle \rho^2 \rangle = \text{const} \quad \left(\langle \rho^i \rangle = \sum_{k=1}^{n} \frac{m_k{}^i}{v^i} \right) \tag{1.2}$$

Defining the quantities $\langle \rho^i \rangle$ (appearing within the brackets in (1.2)) in the following manner:

$$\langle \rho^1 \rangle = \rho \sum_{k=1}^{n} \frac{c_k{}^1}{m}$$

we can show that the equation $\rho = \langle \rho^1 \rangle$ is equivalent to the relation

$$m = \sum_{k=1}^{n} c_k{}^1 \equiv c^1 \tag{1.3}$$

Unlike the concentrations given in (1.1) the true concentrations $\langle c_k{}^i \rangle$ are defined by the formula (no summation is performed over i)

$$\langle c^i \rangle = m_k{}^i / (v^i \rho^i)$$

By virtue of (1.3), the concentrations $c_k{}^i$ and $\langle c_k{}^i \rangle$ are connected by

$$c_k{}^1 = m \langle c_k{}^1 \rangle, \qquad c_k{}^2 = (1 - m) \langle c_k{}^2 \rangle$$

Clearly, the sum of all concentrations $c_k{}^i$ is equal to unity. Here the substances, concentration of which is purposely retained constant, are not grouped separately, but this can easily be performed in the final equations.

The concentration changes in the phases can be caused by the influx of matter, the chemical reactions, and diffusion within the phases. For simplicity we shall assume further that the chemical reactions may take place only in the phase 2. Since the velocities of the components coincide, therefore diffusion which is governed by the difference between the component velocities and the average velocity of the medium, does not appear within the phases.

Taking the above into account we can write the equations of conservation of mass of the components in the form

$$\rho\frac{dc_k{}^1}{dt}=Q_k{}^1, \qquad k=1,2,\ldots,n \tag{1.4}$$

$$\rho\frac{dc_k{}^2}{dt}=Q_k{}^2+\sum_{j=1}^{r}\nu_{kj}J_j, \qquad k=1,2,\ldots,n$$

where $Q_k{}^i$ is the velocity of influx of the kth species into the phase i, $\nu_{kj}J_j$ is the rate of formation of the kth component in the jth chemical reaction and r is the number of reactions. The quantity ν_{kj} is related to the molecular mass M_k of the kth component and is proportional to the stoichiometric coefficient accompanying the component k entering the equation of the jth chemical reaction. The coefficients ν_{kj} are assumed positive when the component k appears in the right-hand side of the equation of reaction (the substance is generated due to reaction) and negative, when it enters the left-hand side of the equation. Since the mass is conserved in each reaction, we have

$$\sum_{k=1}^{n}\nu_{kj}=0, \qquad j=1,2,\ldots,r$$

We further assume that the matter can enter phase 1 only from phase 2 or from the distributed "external" sources, and phase 2 only from phase 1. For this reason we have

$$Q_k{}^1=-Q_k+Q_k^e, \qquad Q_k{}^2=Q_k \tag{1.5}$$

where $Q_k{}^e$ is the source strength in phase 1 (a prescribed quantity). The condition of incompressibility implies that $Q_1{}^e+\ldots+Q_n{}^e=0$.

We note that by virtue of the assumption that the component velocities coincide, the derivatives d/dt in (1.4) have the same meaning for all k and for both phases. Assuming that the medium velocity is small, we shall neglect the convective transport of matter. Then Eqs. (1.4) with (1.5) taken into account, can be written in the form

$$\rho\frac{\partial c_k{}^1}{\partial t}=-Q_k+Q_k^e, \qquad \rho\frac{\partial c_k{}^2}{\partial t}=Q_k+\sum_{j=1}^{r}\nu_{kj}J_j \tag{1.6}$$

Let us write the equations of continuity (with the constancy of density taken into account) and of conservation of momentum for the medium on the whole (the deformations are assumed small)

$$\operatorname{div}\mathbf{u}=0 \tag{1.7}$$

$$\rho\frac{\partial^2 u_i}{\partial t^2}=\frac{\partial p_{ij}}{\partial x_j}+\rho f_i \tag{1.8}$$

Here u_i is the displacement vector, f_i are the mass forces and p_{ij} is the stress tensor for the whole medium. A fixed, orthonormal coordinate system is used in (1. 8) and the subsequent equations.

For each of the phases and for the whole medium we introduce free energy, entropy, temperature, the stress and strain tensors. Further we assume that the temperatures and strains in both phases coincide and that the following relations hold:

$$\begin{aligned} s &= m\langle s^1\rangle + (1-m)\langle s^2\rangle \\ F &= m\langle F^1\rangle + (1-m)\langle F^2\rangle \\ p_{ij} &= m\langle p^1\rangle_{ij} + (1-m)\langle p^2\rangle_{ij} \end{aligned} \tag{1.9}$$

where s, F and T denote the entropy, free energy and temperature, respectively, for the medium on the whole.

Following assumptions are made with regard to the rheological properties of the phases, in accordance with the assertion 4°. Phase 1 is elastic and phase 2 is viscoelastic. For this reason it is expedient to write the strain tensor for the phase 2 in the form of a sum $\varepsilon_{ij} = \eta_{ij} + \Delta_{ij}$, where η_{ij} is the reversible part of the total strain. In accordance with the accepted meaning of the concepts of elasticity and viscoelasticity we define for each of the phases the free energy, and postulate the Gibbs relation

$$m\langle F^1\rangle = F_0^1(c^1_k, T) + \frac{1}{2\rho} k_{ijlm}\varepsilon_{ij}\varepsilon_{lm} \tag{1.10}$$

$$d\langle F^1\rangle = \frac{1}{\rho}\langle p^1\rangle_{ij} d\varepsilon_{ij} - \langle s^1\rangle dT + \sum_{k=1}^{n} \langle \mu_k^1\rangle d\langle c_k^1\rangle \tag{1.11}$$

$$(1-m)\langle F^2\rangle = F_0^2(c_k^2, T) + \frac{1}{2\rho} m_{ijlm}\eta_{ij}\eta_{lm} \tag{1.12}$$

$$d\langle F^2\rangle = \frac{1}{\rho}\langle p^2\rangle_{ij} d\eta_{ij} - \langle s^2\rangle dT + \sum_{k=1}^{n} \langle \mu_k^2\rangle d\langle c_k^2\rangle \tag{1.13}$$

$$\langle \mu_k^i\rangle = \partial\langle F^i\rangle / \partial\langle c_k^i\rangle, \qquad \langle s^i\rangle = \partial\langle F^i\rangle / \partial T$$

Here $\langle \mu_k^i\rangle$ denotes the chemical potential of the kth component of the ith phase. The strains and displacements are assumed small, consequently we have (with the incompressibility of the medium taken into account)

$$\varepsilon_{ii} = 0, \quad \eta_{ii} = \Delta_{ii} = 0 \tag{1.14}$$

Taking into account the presence of the mass sources in phase 1 and the definitions (1. 9) – (1. 14), we can write the equation of the heat influx as well as the expression for the second law of thermodynamics, in the form

$$dF = \frac{1}{\rho} p_{ij} d\varepsilon_{ij} - s dT - dq' + \frac{1}{\rho} \sum_{k=1}^{n} \frac{\partial F}{\partial c_k^1} Q_k^e dt \tag{1.15}$$

$$T ds = dq^e + dq' + \sum_{k=1}^{n} T S_k^1 \frac{Q_k^e}{\rho} dt \tag{1.16}$$

$$S_k^1 = \left(\frac{\partial s}{\partial c_k^1}\right)_{T, c_l^i (l\neq k), \eta_{ij}, \varepsilon_{ij}} = \frac{\partial^2 F}{\partial T \partial c_k^1}$$

Here the expression for partial entropy S_k^1 was derived employing the assumptions made earlier about the free energy.

2. Rheological relations. The system (1.6) – (1.10) is not closed. It contains the following unknown quantities Q_k, J_i and p_{ij}, for which we require defining equations.

The formulas (1.10) and (1.12) for the free energy of the phases and the Gibbs equations (1.11) and (1.13) directly yield expressions for the deviators of the stress tensors in the phases 1 and 2

$$\sigma_{ij}^1 = k_{ijlm}\varepsilon_{lm}, \qquad \sigma_{ij}^2 = m_{ijlm}\eta_{lm} \tag{2.1}$$

$$\sigma_{ij}^\alpha = c^\alpha (\langle p^\alpha\rangle_{ij} - {}^1/_3 \langle p^\alpha\rangle_{ll}\delta_{ij}) \tag{2.2}$$

$$(c^1 = m,\ c^2 = (1-m),\ \alpha = 1,\ 2)$$

Taking into account the identities

$$\partial c^\alpha \langle F^\alpha\rangle / \partial c_k^\alpha = \langle \mu_k^\alpha\rangle + \langle F^\alpha\rangle - \sum_{l=1}^{n} \langle \mu_l^\alpha\rangle\langle c_l^\alpha\rangle, \qquad \alpha = 1,2$$

and introducing the chemical potentials of the phases 1 and 2 related to the unit mass of the whole medium

$$\mu_k^\alpha = \frac{\partial c^\alpha \langle F^\alpha\rangle}{\partial c_k^\alpha}, \qquad \alpha = 1,2$$

we transform the Gibbs equations (1.13) and (1.15) to the form

$$\begin{aligned} d(c^1\langle F^1\rangle) &= \frac{1}{\rho} c^1 \langle p^1\rangle_{ij} d\varepsilon_{ij} - c^1 \langle s^1\rangle dT + \sum_{k=1}^{n} \mu_k^1 dc_k^1 \\ d(c^2\langle F^2\rangle) &= \frac{1}{\rho} c^2 \langle p^2\rangle_{ij} d\eta_{ij} - c^2 \langle s^2\rangle dT + \sum_{k=1}^{n} \mu_k^2 dc_k^2 \end{aligned} \tag{2.3}$$

We use the relation $\rho dq^{(e)} = -\operatorname{div} \mathbf{q}\, dt$ to define the heat flux vector $\mathbf{q}$. Then (1.15) and (1.16) with (2.3) taken into account we obtain the following equation of the entropy balance:

$$\rho\frac{ds}{dt} - \frac{\partial s}{\partial c_k^1} Q_k^e = -\operatorname{div}\frac{\mathbf{q}}{T} + \frac{R}{T} \tag{2.4}$$

$$R = -\frac{\mathbf{q}}{T}\operatorname{grad} T + \sigma_{ij}^2 \Delta_{ij}^{\boldsymbol{\cdot}} + \sum_{k=1}^{n} Q_k(\mu_k^1 - \mu_k^2) - \sum_{j=1}^{r} J_j A_j \tag{2.5}$$

$$A_j = \sum_{k=1}^{n} \nu_{kj}\mu_k^2, \qquad j = 1, 2, \ldots r \tag{2.6}$$

Here R is a dissipative function and the quantity A_j denotes, by definition, the affinity of the jth chemical reaction.

We shall assume that the generalized fluxes $\mathbf{q}$, σ_{ij}^2, Q_k and J_j can be expressed in a linear manner, in terms of the generalized thermodynamic forces

$$-\frac{1}{T^2}\operatorname{grad} T, \quad \frac{\Delta_{ij}^{\boldsymbol{\cdot}}}{T}, \quad \frac{\mu_k^1 - \mu_k^2}{T}, \quad -\frac{A_j}{T}$$

By virtue of the assertion 4° of Sect. 1 referring to the anisotropy of the medium, the coefficients appearing in the above relations as well as the coefficients k_{ijlm} and m_{ijlm} in (2.1) are determined by the metric tensor g_{ij} and the tensors characterizing the symmetry of the medium, and the Curie principle does not hold in the general case.

We now assume that transversal isotropy exists for the medium under consideration and that the following symmetry transformations are allowed: one (rotational) axis of infinite order, an infinite number of transverse axes of second order (180° rotations), an infinite number of longitudinal symmetry planes, one transverse symmetry plane and a center of symmetry. It was shown in [2] that this symmetry group, denoted by $m \cdot \infty : m$, is characterized by the metric tensor g_{ij} and one additional tensor b_{ij}. In the ortonormalized coordinate system in which the basis vector $\mathbf{e}_3$ is directed along the rotational axis of infinite order, all components of the tensor b_{ij} except $b_{33} = 1$ are equal zero. The general expression for arbitrary tensor functions $\mathbf{B} = \mathbf{B}(\mathbf{g}, \mathbf{b})$ depending only on the above two arguments, is [2]

$$B_i = 0, \quad B_{ij} = \beta_1 \delta_{ij} + \beta_2 b_{ij}, \quad B_{ijk} = 0 \tag{2.7}$$

The fourth rank tensor B_{ijkl} is characterized, generally speaking, by ten independent parameters. We shall write an expression for this tensor with additional conditions of symmetry taken into account. Taking into consideration the relations (2.7) we shall write, in the general form, the allowed linear relationships connecting the generalized fluxes and generalized forces

$$q_i = -\alpha_{ij} \frac{\partial T}{\partial x_j} \tag{2.8}$$

$$\sigma_{ij}^2 = \beta_{ijkl} \Delta_{kl}^{\cdot} - \sum_{\beta=1}^{r} \lambda_{\beta ij} A_\beta + \sum_{\alpha=1}^{n} s_{\alpha ij} (\mu_\alpha^1 - \mu_\alpha^2)$$

$$J_\beta = \Lambda_{\beta ij} \Delta_{ij}^{\cdot} - \sum_{\gamma=1}^{r} k_{\beta\gamma} A_\gamma + \sum_{\alpha=1}^{n} r_{\beta\alpha} (\mu_\alpha^1 - \mu_\alpha^2)$$

$$Q_\alpha = S_{\alpha ij} \Delta_{ij} - \sum_{\beta=1}^{r} r_{\alpha\beta} A_\beta + \sum_{\varkappa=1}^{n} n_{\alpha\varkappa} (\mu_\varkappa^1 - \mu_\varkappa^2)$$

The factor containing temperature is included in the appropriate coefficients. In the above equations $k_{\beta\gamma}$, $r_{\beta\alpha}$, $r_{\alpha\beta}$ and $n_{\alpha\varkappa}$ are scalars; α_{ij}, $\lambda_{\beta ij}$, $\Lambda_{\beta ij}$, $s_{\alpha ij}$ and $S_{\alpha ij}$ are second rank tensors and β_{ijkl} is a fourth rank tensor (for clarity, here and in the following, the nontensorial indices are denoted by Greek letters).

The Onsager relations impose the following restrictions on the coefficients:

$$\lambda_{\beta ij} = -\Lambda_{\beta ij}, \qquad s_{\alpha ij} = -S_{\alpha ij}$$

$$r_{\alpha\beta} = r_{\beta\alpha}, \qquad k_{\beta j} = k_{j\beta}, \qquad n_{\alpha\varkappa} = n_{\varkappa\alpha}, \qquad \alpha_{ij} = \alpha_{ji}, \qquad \beta_{ijkl} = \beta_{klij}$$

The tensor coefficients entering (2.1) and the expression for σ_{ij}^2 in (2.8) must satisfy additional relations emerging from the requirement that the stress tensors be symmetric and from the fact that σ_{ij}^1 and σ_{ij}^2 are, by definition, deviators.

For the fourth rank tensors B_{ijkl} (i.e. for k_{ijlm}, m_{ijkl} and β_{ijkl}) the following relations must hold

$$B_{ijkl} = B_{jikl}, \qquad B_{ijkl} = B_{klij}, \qquad B_{iikl} = 0 \tag{2.9}$$

The general expression for the fourth rank tensor $\mathbf{B} = \mathbf{B}(\mathbf{g}, \mathbf{b})$ satisfying the conditions (2.9) under the transversal isotropy, has the form

$$B_{ijkl} = \lambda_1 (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - {}^2/_3\, \delta_{ij}\delta_{kl}) + \lambda_2 (\delta_{ij} b_{kl} + \delta_{kl} b_{ij} - {}^1/_3 \delta_{ij}\delta_{kl} - 3 b_{ij} b_{kl}) +$$
$$\lambda_3 (\delta_{ik} b_{jl} + \delta_{jk} b_{il} + \delta_{il} b_{jk} + \delta_{jl} b_{ik} - 4 b_{ij} b_{kl})$$

The second rank tensors $\Lambda_{\beta ij}$ and $s_{\alpha ij}$ must be symmetric and have a zero trace. Therefore from (2.7) follows:

$$\Lambda_{\beta ij} = \Lambda_\beta (\delta_{ij} - 3b_{ij}), \qquad s_{\alpha ij} = s_\alpha (\delta_{ij} - 3b_{ij})$$
$$(\alpha = 1, 2, \ldots, n; \ \beta = 1, 2, \ldots, r)$$

The defining relations (2.1) and (2.8) contain, apart from two thermal conductivity coefficients, $9 + n + 2nr + {}^1/_2\,(n^2 + r^2 + n + r)$ independent scalar quantities (including $nr - r$ stoichiometric coefficients).

In a specific case when the mass transfer between the phases is determined by the chemical potentials only and does not depend on $\dot{\Delta}_{ij}$ and A_β (at the same time the reaction rates and the stresses in phase 2 do not depend explicitly on the chemical potentials of phase 1), we must set $s_{\alpha ij} = 0$ and $r_{\alpha ij} = 0$, whereupon Eqs. (2.8) become

$$\sigma_{ij}^2 = \beta_{ijkl}\Delta_{kl}^{\cdot} + \sum_{\beta=1}^{r} \Lambda_{\beta ij} A_\beta \tag{2.10}$$

$$J_\beta = \Lambda_{\beta ij}\Delta_{ij}^{\cdot} - \sum_{\gamma=1}^{r} k_{\beta\gamma} A_\gamma \tag{2.11}$$

$$Q_k = \sum_{\varkappa=1}^{n} n_{\alpha\varkappa} (\mu_\varkappa^1 - \mu_\varkappa^2) \tag{2.12}$$

After this simplification the number of the scalar parameters given above is reduced to $9 + nr + {}^1/_2\,(n^2 + r^2 + n + r)$. The condition of positive definiteness of the dissipative function R imposes restrictions on the parameters in the defining equations. In particular, we have

$$n_{\alpha\alpha} > 0, \quad n_{11}n_{22} - n_{12}^2 > 0$$

The rheological equation connecting the deviator $p_{ij}^d = \sigma_{ij}^1 + \sigma_{ij}^2$ of the total stress tensor with the total strain ε_{ij} is obtained by eliminating Δ_{ij}° from (2.10) in accordance with the formula which follows from (2.1)

$$\Delta_{ij} = \varepsilon_{ij} - \overset{\circ}{m}_{klij}(p_{kl}^d - k_{klnm}\varepsilon_{nm}) \tag{2.13}$$

Here m_{klij}° is the inverse of m_{ijkl}, i.e. $m_{klij}^\circ\, m_{ijpq} = \delta_{kp}\delta_{lq}$. Performing the computations with (1.11) and (2.13) taken into account we find

$$A_{ijkl}p_{kl}^{\cdot d} + p_{ij}^d = k_{ijkl}\varepsilon_{kl} + D_{ijkl}\varepsilon_{kl}^{\cdot} + N_{ij} \tag{2.14}$$
$$A_{ijkl} = \beta_{ijnm}\overset{\circ}{m}_{klnm}$$
$$D_{ijkl} = \beta_{ijkl} + \beta_{ijnm}\overset{\circ}{m}_{nmpq}k_{pqkl}$$
$$N_{ij} = \sum_{\beta=1}^{r} \Lambda_{\beta ij}A_\beta = \sum_{\beta=1}^{r}\sum_{\alpha=1}^{n} \Lambda_{\beta ij}\nu_{\alpha\beta}\frac{\partial F}{\partial c_\alpha^{\,2}}$$

Using (2.13) to eliminate Δ_{ij} from (2.11), we obtain

$$J_\beta = R_{\beta ij}\varepsilon_{ij} + P_{\beta ij}p_{ij}^{\cdot d} - \sum_{\gamma=1}^{r} k_{\beta\gamma}A_\gamma$$
$$R_{\beta ij} = \Lambda_{\beta ij} + \Lambda_{\beta nm}\overset{\circ}{m}_{nmpq}k_{pqij} \tag{2.15}$$
$$P_{\beta ij} = -\Lambda_{\beta nm}\overset{\circ}{m}_{nmij}$$

We note that the formulas (2.14) and (2.15) are written under the simplifying assumption that all coefficients in the defining equations are constant.

Thus, the system describing motion of the continuous medium under consideration, possessing mechanochemical reactions, contains the following equations: (1) equation of continuity (1.7); (2) equation of conservation of momentum (1.8) with the rheological equation (2.14); (3) equation of conservation of mass of the components (1.6) together with the defining equations (2.12) and (2.15); (4) equation of heat influx. The last equation is obtained from (2.4) by inserting into it $s = -\partial F / \partial T$ and the explicit expressions for R and $\mathbf{q}$. When the coefficients in the defining equations are constant, this equation becomes

$$\varphi(T, c_\beta^1, c_\beta^2)\frac{dT}{dt} = \frac{\alpha_{ij}}{\rho}\frac{\partial^2 T}{\partial x_i \partial x_j} + \psi(p_{ij}^{\cdot d}, \varepsilon_{ij}^{\cdot}, T, c_\beta^1, c_\beta^2)$$

$$\varphi = -T\frac{\partial^2 F}{\partial T^2} \qquad (2.16)$$

$$\psi = T\sum_{\beta=1}^{n}\left[\frac{\partial^2 F}{\partial T \partial c_\beta^1}\left(\frac{dc_\beta^1}{dt} - \frac{1}{\rho}Q_\beta^e\right) + \frac{\partial^2 F}{\partial T \partial c_\beta^2}\frac{dc_\beta^2}{dt}\right] +$$

$$\frac{1}{\rho}\beta_{ijkl}\,(C_{ijnm}\dot{\varepsilon}_{nm} - \overset{\circ}{m}_{ijpq}\dot{p}_{pq}^{d})(C_{klrs}\varepsilon_{rs}^{\cdot} - \overset{\circ}{m}_{klab}\dot{p}_{ab}^{d}) +$$

$$\frac{1}{\rho}\sum_{\beta=1}^{n}\sum_{\varkappa=1}^{n}\left(\frac{\partial F}{\partial c_\beta^1} - \frac{\partial F}{\partial c_\beta^2}\right)\left(\frac{\partial F}{\partial c_\varkappa^1} - \frac{\partial F}{\partial c_\varkappa^2}\right)n_{\beta\varkappa} +$$

$$\frac{1}{\rho}\sum_{\beta,\gamma=1}^{r}\sum_{\alpha,\varkappa=1}^{n}k_{\beta\gamma}\nu_{\alpha\beta}\nu_{\varkappa\gamma}\frac{\partial F}{\partial c_\alpha^2}\frac{\partial F}{\partial c_\varkappa^1} \qquad (C_{ijnm} = \delta_{in}\delta_{jm} + \overset{\circ}{m}_{pqij}k_{pqnm})$$

We close the system by specifying the free energy of the medium on the whole and the coefficients in the defining equations.

3. Discussion of results. The model constructed has a number of properties characteristic for the muscular tissue (see [5 – 8]). Thus, from Eqs. (2.14) it follows that in absence of strains in the medium, the active stresses governed by the chemical reactions in phase 2, may exist and, conversely, the medium may undergo active deformation in the absence of stresses. Relations (2.8) show that the mechanical processes may exert a reciprocal influence on the course of the chemical reaction and on the transfer of the chemical components between the phases. We note that the amount of energy consumed during contraction depends on the character of the mechanical processes taking place within the medium. In particular, energy may be consumed and heat released even when the state of mechanical equilibrium prevails and no mechanical work is being done.

The commencement of the process of development of active stresses and strains and the beginning of contraction in the muscular tissue is governed by a shift in the chemical equilibrium which may be caused, e.g. by the change in the interphase permeability coefficients $n_{\alpha\varkappa}$ in (2.12). This formula also admits the existence of so called active transfer (from the phase with lower concentration to the phase with higher concentration) of certain reagents, while other reagents diffusing in the "normal" manner play the part of chemically active carriers.

One-dimensional models of the muscle known at present [13, 14, 16 – 19] are based on an empirical relation (Hill's equation) connecting the velocity of contraction of the "shortening element" with the stress developed by it. In our model Eq. (2.10) serves as an analog of such a relation; the velocity of contraction corresponds to the quantity Δ_{33} and the stress developed in it to the quantity $\sigma_{33}{}^2$. Since the second term in the right-hand side of (2.10) may indirectly depend on the character of the contraction process (e. g. on whether the contraction takes place at a constant strain or a constant stress), the above relation will be different for the different processes.

At present it is assumed proved that the contraction of muscular tissue is accompanied by splitting of one of the acids (adenosinetriphosphoric acid) and the free energy released during this process is estimated to be the basic source of energy for the contraction [5 – 7]. The model which we have constructed, conforms to this assertion. Indeed, from (1.9) – (1.15) it follows that any irreversible processes in an isolated volume of the medium, and this also includes the processes during which mechanical work is done, take place at the expense of the free energy as a result of chemical processes. For the well known models of contractile tissue which do not directly utilize the concept of chemical reactions, the free energy in the equation for the heat influx

$$dF = \frac{1}{\rho} p_{ij} d\varepsilon_{ij} - s\, dT - dq' + dq^{**} \qquad (3.1)$$

is specified as a function of the strain tensor and temperature (as before), as well as of certain "internal strain" tensor $\varkappa_{ij}$ (characterizing the internal degrees of freedom [4]). Moreover, in (3.1) the influx of external nonmechanical energy dq^{**} is also given.

Let us now define F and dq^{**} so as to arrive at the rheological equation (0.1) for a bioelastic body [12] and let us establish the connection between dq^{**} and the change in free energy caused by the chemical processes for the model given in Sects. 1 and 2, for $\beta_{ijkl} = 0$ (when $\beta_{ijkl} = 0$, neither the rheological equation (2.4) nor (0.1) contain derivatives with respect to time).

Let us expand the free energy into a series in parameters ε_{ij} and $\varkappa_{ij}$, retaining the linear and quadratic terms only and assuming that the coefficients of expansion are independent of temperature. The medium is assumed incompressible. By definition the bioelastic body is in a passive state if $dq^{**} = dq' = 0$. Then from (3.1) we obtain the rheological equation of the conventional linear theory of elasticity

$$\varepsilon_{ij} = f_{ij}(\varkappa_{lm}) = M_{ijlm}\varkappa_{lm} + C_{ij}$$
$$p_{ij}{}^{d} = A_{ijkl}\varepsilon_{kl} + B_{ijkl}\varkappa_{kl} = k_{ijlp}\varepsilon_{lp}$$

We assume that the contraction is governed by an influx of a nonmechanical energy $dq^{**} \neq 0$, such, that

$$dq^{**} = dq' + \Omega_{ij}\, d\varkappa_{ij}$$

Then from (3.1) we can obtain an expression for Ω_{ij} and the following rheological equation for a bioelastic body in an active state:

$$p_{ij}{}^{d} = k_{ijlm}\varepsilon_{lm} + n_{ijkl}(\varepsilon_{kl} - f_{kl}) \qquad (3.2)$$

which coincides with (0.1) within the accuracy of the notation used. Clearly, there is a correspondence between the rheological equation (3.2) and the rheological equation (2.14) or (2.10), provided that

$$\beta_{ijkl} = 0 \qquad (3.3)$$

The expression for the energy influx dq^{**} with (3.2) and the expression for Ω_{ij} both taken into account, assumes the form

$$dq^{**} = dq' - \rho^{-1} n_{ijkl} (\varepsilon_{ij} - f_{ij}) = dq' - \rho^{-1} (p_{ij}{}^{d} - k_{ijlm} \varepsilon_{lm}) \, df_{ij} \qquad (3.4)$$

Then the change in the free energy for the model of the medium constructed in Sects. 1 and 2 resulting from the chemical reactions ($\beta_{ijkl} = 0$) is

$$\sum_{k=1}^{n} \left(\frac{\partial F}{\partial c_k{}^1} + \frac{\partial F}{\partial c_k{}^2} \right) \Bigg|_{Q_k{}^e = 0} = - dq' + \frac{1}{\rho} (p_{ij}{}^{d} - k_{ijkl} \varepsilon_{kl}) \, d(\varepsilon_{ij} - \eta_{ij}) \qquad (3.5)$$

Equating (3.4) and (3.5) with (3.2), (3.3) and (2.1) taken into account, we obtain

$$dq^{**} = - \sum_{k=1}^{n} \left(\frac{\partial F}{\partial c_k{}^1} \, dc_k{}^1 + \frac{\partial F}{\partial c_k{}^2} \, dc_k{}^2 \right), \qquad Q_k{}^e = 0$$

Thus, dq^{**} is equal to the change in free energy due to chemical reactions taken with the opposite sign.

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ELECTRIC FIELD IN A MHD CHANNEL OF RECTANGULAR CROSS SECTION IN THE PRESENCE OF THE HALL EFFECT

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The present paper deals with the spatial distribution of the electrostatic potential in a channel with two electrodes in the presence of the Hall effect. The velocity profile is inhomogeneous and corresponds to the velocity diminishing down to zero at the channel walls. The problem of determining the electric field in the channel is reduced to that of solving a boundary value problem with mixed boundary conditions for an elliptic type equation. One of the versions of the Wiener-Hopf method is used in the course of solution.

The three-dimensional distribution of the electric field in a MHD channel has been studied, because of considerable difficulties of mathematical nature encountered, only for the simplest cases of isotropically conducting media, i. e. for the cases when the walls have uniform conducting properties, or when an electrode zone is present in the channel [1 – 7]. For the anisotropic conductivity of the medium only plane problems have been studied [8, 9].

1. The case of semi-infinite electrodes. 1°. Let us consider a flow of a viscous, incompressible, anisotropically conducting medium in a MHD channel of rectangular cross section $|x| < \infty$, $|y| < b$, $|z| < 1$, in an external homogeneous magnetic field $\mathbf{H}_0$ $(0, H_0, 0)$, $H_0 =$ const. For $y = \pm b$, the channel walls are insulators, while the other two walls $(z = \pm 1)$ are insulators for $x < 0$ and perfectly conducting electrodes for $x > 0$. The velocity of the medium is